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Network formation determined by the diffusion process of random walkers

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Abstract

We studied the diffusion process of random walkers in networks formed by their traces. This model considers the rise and fall of links determined by the frequency of transports of random walkers. In order to examine the relation between the formed network and the diffusion process, a situation in which multiple random walkers start from the same vertex is investigated. The difference in diffusion rate of random walkers according to the difference in dimension of the initial lattice is very important for determining the time evolution of the networks. For example, complete subgraphs can be formed on a one-dimensional lattice while a graph with a power-law vertex degree distribution is formed on a two-dimensional lattice. We derived some formulae for predicting network changes for the 1D case, such as the time evolution of the size of nearly complete subgraphs and conditions for their collapse. The networks formed on the 2D lattice are characterized by the existence of clusters of highly connected vertices and their life time. As the life time of such clusters tends to be small, the exponent of the power-law distribution changes from $\gamma \simeq 1$–2 to $\gamma \simeq 3$.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The description of the essential features of real networks has been of great interest since the recognition of the so-called complex networks. The properties of complex networks such as small-world phenomena and scale-free vertex degree distribution can often be found in various systems and have an important role to fulfil their functions. Therefore various systems such as social networks, biology systems and networks in the Internet can be studied by unified methods based on the structure of complex networks (for reviews see [1–4]).

In network modeling, preferential attachment of new nodes to highly connected nodes already existing in the network [5] is simple, but it is a progressive principle beyond random linking of vertices already existing in a graph. Growing networks by attachment of new nodes and constraints such as ageing and attractiveness of nodes are key frameworks for considering network modeling [6–11].

On the other hand, it was found recently that the use of random walkers to determine where new nodes attach can provide not only global (scale free, small mean vertex–vertex distance) but also local structures of networks (strong clustering, community, assortative mixing) [12–14]. It is significant to elucidate the mechanism to reproduce local properties of networks [15, 16], because simple attachment of nodes hardly leads to such local properties. Furthermore, consideration of local structure is necessary to describe real networks, for the existence of community and assortative mixing [17–20] are found to be characteristics of real networks such as social networks. Note that movements of random walkers are restricted by local connections between nodes, but do not need complete information on the network. This feature is consistent with our experiences that we tend to take action based only on the information around ourselves.

In our previous paper, we proposed a time evolving network model, which considers the rise and fall of links determined by the frequency of transportation of random walkers [21]. A characteristic of this model is that new connections between vertices already existing in a regular lattice are stimulated by transportation between the lattice points without the attaching of new vertices to the graph. This model is an extreme one that ignores growing effects by the attaching of new vertices to networks. Nevertheless, broad degree distributions and the small-world phenomenon can be found in this model. The detailed description of the model is as follows (see figure 1).

For each discrete time t $(t = 0, 1, 2, \ldots)$ random graph $\{G_t\}$ is transformed into $\{G_{t+1}\}$ by simultaneous movement of w independent random walkers on $\{G_t\}$, where $\{G_0\}$ is a one or two-dimensional squared lattice. Each $\{G_t\}$ consists of original lattice $\{G_0\}$ and edges joining a pair of vertices in $\{G_0\}$. Each edge is associated with integer 'strength' that evolves with time. At initial time, each strength of edges is 1. Added edges and strength of their edges are regulated by the following two rules. First: at each time step t, each random walker moves with equal probability from a vertex the walker currently stays to one of directly connected vertices by edges in $\{G_t\}$. If the nth walker's location sequence is $\left\{\mathbf{x}_t^{(n)}\right\}$, then the movement $\left\{\mathbf{x}_t^{(n)}\right\} \longrightarrow \left\{\mathbf{x}_{t+1}^{(n)}\right\}$ results in increase of strength of the edge joining between $\left\{\mathbf{x}_t^{(n)}\right\}$ and $\left\{\mathbf{x}_{t+1}^{(n)}\right\}$, and the creation of a new edge of strength 1 between $\left\{\mathbf{x}_{t-1}^{(n)}\right\}$ and $\left\{\mathbf{x}_{t+1}^{(n)}\right\}$. If there is already a edge between $\left\{\mathbf{x}_{t-1}^{(n)}\right\}$ and $\left\{\mathbf{x}_{t+1}^{(n)}\right\}$, its strength is increased by 1 (figure 1(*a*)). Second: after adding of edges and strengthen of edges at each step explained above, the strengths of all edges are independently decreased by 1 with probability p_{d}, and edges that attain strength 0 are removed except edges existing in the original squared lattice $\{G_0\}$. Edges of strength 1 included in $\{G_0\}$ do not lose their strengths any further. One example of the graph formed by the rule is indicated in figure 1(*b*).

In this model, the rise and fall of links determined by random transports in the system are represented by strengths of edges, and the exception rule for edges included in $\{G_0\}$ is a simple consideration for the geographical relation between elements in the system, which has rarely been studied up to now. Nearest neighboring vertices in $\{G_0\}$ are always joined for each t, while edges joining vertices far from each other in $\{G_0\}$ can exist at least only after a random walker visits the two vertices accidentally.

While the topology of $\{G_t\}$ attracts our interest, dynamical behavior with the movement of random walkers and extinction of edges has a value to be studied. So far, however, we have

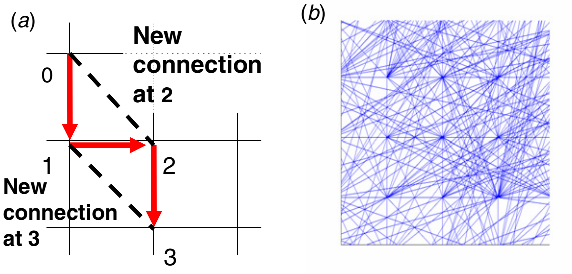

Figure 1. (*a*) New connections between vertices in a regular lattice. If a random walker moves in the order vertex 0, 1, 2 and 3, vertices 0 and 2 are newly joined at second time step and vertices 1 and 3 are newly joined at third time step. The strength of edges where the walker has passed increases by 1. Multiple random walkers are independent of each other, and can pass newly created edges (dashed lines). (*b*) One example of a part of the graph formed by the rule.

been concentrating on the topology of $\{G_t\}$ originating only from a one-dimensional lattice. Because it is natural to expect that highly connected vertices are created by the frequent visit of random walkers to the vertices, the movement of random walkers should be focused on for a more detailed study of the evolution of networks. In the present paper, in order to study multiplier effects of a number of random walkers, we focused on a special case in which all the walkers depart from the same vertex in a one-dimensional or two-dimensional squared lattice, while in the previous work they departed from randomly spread sites in a one-dimensional lattice with the periodic boundary condition. This paper considers a special diffusion problem, but it also takes up the interesting problem of the relation between network topology and diffusion process of the walkers that determine the rise and fall of links.

This paper is organized as follows. In section 2, some typical results for one-dimensional cases are presented in order to highlight the characteristic points of our problem; transition from stable structure when extinction probability $p_{\mathrm{d}} = 0$ to collapse of the structure corresponding to the increase of p_{d}. The structure formed when $p_{\mathrm{d}} = 0$ differs according to the dimension. Section 3 explains two-dimensional cases that exhibit degree distribution with power law when $p_{\mathrm{d}} = 0$. Network structures when $p_{\mathrm{d}} = 0$ described in sections 2 and 3 can be maintained as long as an initial time interval is concerned especially when p_{d} is small. Sections 4 and 5 discuss conditions for such maintenance of network structure of one-dimensional and two-dimensional cases, respectively. Sufficiently large p_{d} free random walkers from the network formed around their starting point. This situation results in the emigration of the networks. Section 6 examines the process leading to such an emigration by the observation of the life time of highly connected vertices and number of vertices within a few steps from the highest connected vertex.

2. Observation of one-dimensional case

In this section, some aspects of one-dimensional (1D) case are explained in order to describe the characteristics of this type of network evolution. First, it should be noted that even the movement of one walker is restricted by edges created by the past movement of the walker. As an example of such a restriction, let us take the case where extinction probability p_{d} is 0, which means that the number of edges monotonously increases with time.

In figures 2(*a*) and (*b*), changes in the location of a walker with time and spatial distribution of vertex degree on the 1D lattice are presented, respectively. Random walkers started from the origin in the 1D lattice and wandered around the starting point in figure 2. Figures 2(*a*)

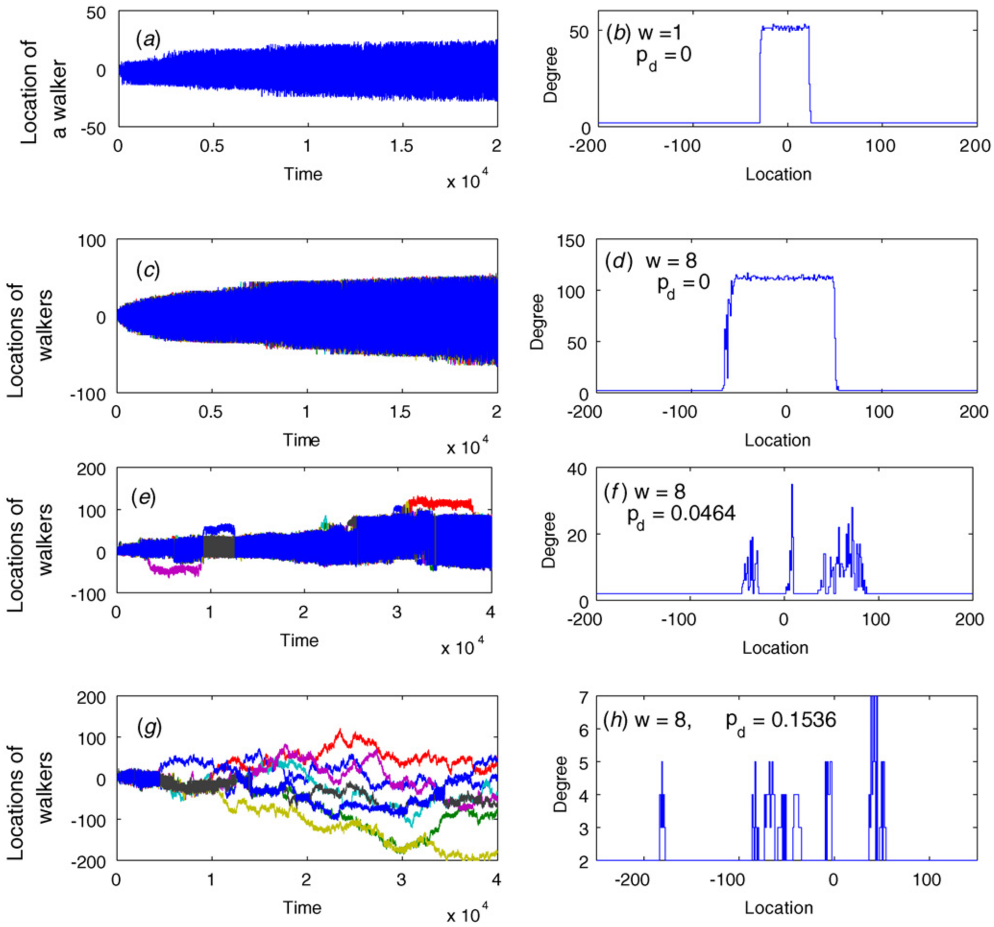

Figure 2. Movements of random walkers. All walkers start from the origin. (*a*) One walker movement ($0 < t < 20\,000$) when $p_\mathrm{d} = 0$. (*b*) Spatial distribution of vertex degree at $t = 20\,000$ when $p_\mathrm{d} = 0$. Most vertices which the walker can reach have the same degree as the number of vertices the walker can reach. (*c*) Eight walkers' movement when $p_\mathrm{d} = 0$ ($0 < t < 20\,000$). (*d*) Spatial distribution of vertex degree formed by eight walkers at $t = 20\,000$ when $p_\mathrm{d} = 0$. (*e*) Eight walkers' movement when $p_\mathrm{d} = 0.0464$ ($0 < t < 40\,000$). (*f*) Spatial distribution of vertex degree formed by eight walkers at $t = 40\,000$ when $p_\mathrm{d} = 0.0464$. (*g*) Eight walkers' movement when $p_\mathrm{d} = 0.1536$ ($0 < t < 40\,000$). (*h*) Spatial distribution of vertex degree formed by eight walkers at $t = 40\,000$ when $p_\mathrm{d} = 0.1536$.

and (*b*) show the formation of a nearly complete subgraph after the wandering of a random walker, as vertices that the walker can stay are joined to almost all vertices the walker has visited in the past.

The walker can move randomly in the nearly complete subgraph but hardly escape from it, because the vertices at the boundary of the nearly complete subgraph have many edges incident to vertices in the nearly complete subgraph, but have only about one edge incident to the vertices outside the nearly complete subgraph. This behavior of the walker reminds us of free particles in a potential well, where we can observe particles in the potential well but cannot find them outside the well. For this reason, we sometimes call subgraphs formed by the wandering of random walkers in a certain area 'potential well', although this

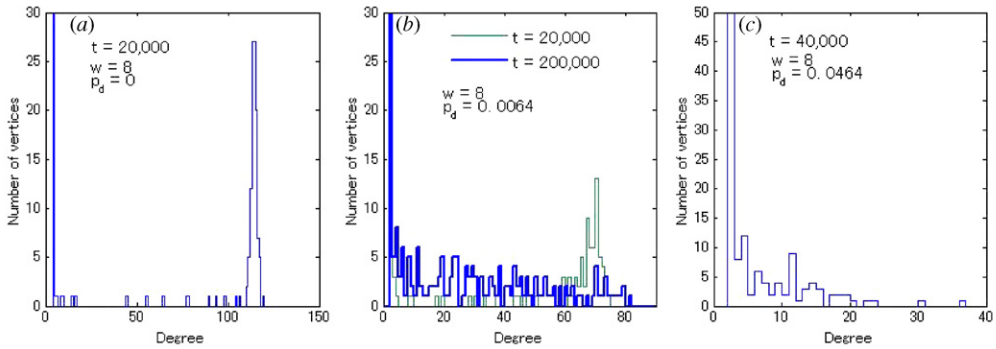

Figure 3. Frequency distributions of vertex degree for the 1D case. (*a*) Distribution for a nearly complete subgraph. (*b*) Broad distribution which is observed after very slow collapse of a complete subgraph. (*c*) Distribution which is observed after easy collapse of the complete subgraph.

is introduced only for an easy explanation of the situation. The potential barrier moves to the outside slowly, since rare escapes from the well would surely make new edges on the outside of the well.

Another interesting aspect of the movement of random walkers is their interaction. Of course they have no direct attractive or repulsive forces, but movements of walkers affect each other via edges created by themselves. According to the analogy to free particles in a potential well, it is natural to expect that more than one random walker are confined to the same potential well. Moreover, the potential well (nearly complete subgraph) has probably widened and deepened with the gathering of random walkers in the potential well. An example of random walkers' behavior in a widened potential well is illustrated in figures 2(*c*) and (*d*). These figures show that random walkers are confined to a complete subgraph widened by themselves. However, the size of nearly complete subgraph is not proportional to the number of random walkers.

Nearly complete subgraphs can be stabilized only when p_d is small. As p_d increases, it becomes difficult to maintain the nearly complete subgraph, and broad vertex degree distributions (see figure 3) can be observed owing to continuous extinction of edges in the subgraph. Further increase of p_d leads to easy escapes of random walkers from the potential well. Such a situation is illustrated in figures 2(*e*) and (*f*), in which a temporal escape of a random walker from the potential well can be observed. Finally, the increase of p_d breaks the potential well (figures 2(*g*) and (*h*)). In this stage, the subgraph cannot be localized in a certain area and begin to emigrate in the 1D lattice corresponding to the near-independent movements of random walkers.

Figure 3 shows the frequency distributions of vertex degree corresponding to each case. From the viewpoint of study of complex networks, observation of an inequality of vertex degree as seen in figures 3(*b*) and (*c*) is more interesting than that of nearly complete subgraph as seen in figure 3(*a*). Some cases need a very long time to transform a complete subgraph into a graph with broad degree distribution (for example, figure 3(*b*)), because random walkers can hardly escape from the potential well as long as the complete subgraph is maintained. On the other hand, when p_d is large enough to permit emigration of subgraphs, maximum degree is noticeably decreased. In this way, the network structure is deeply related to the diffusion behavior of random walkers.

In the following sections, some problems prompted by the overview of the 1D case are examined numerically and theoretically. First, the next section explains the network structure

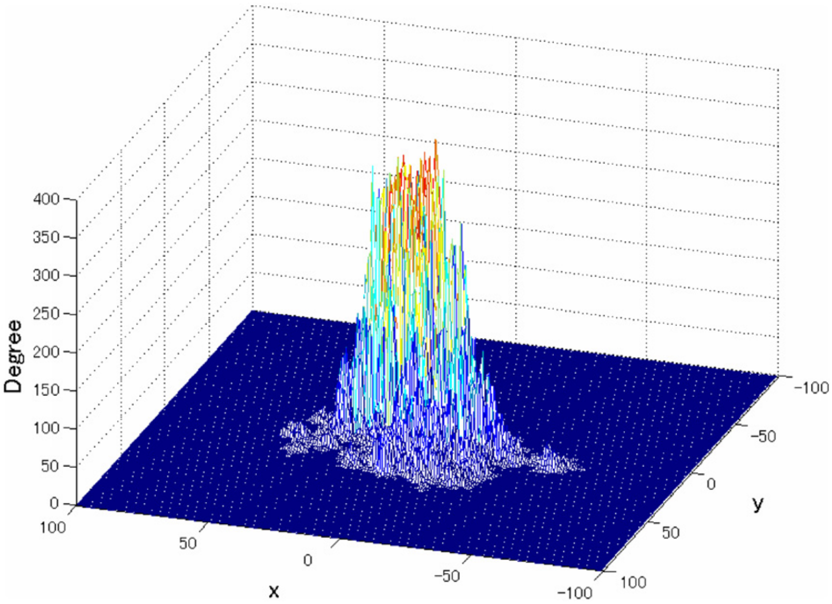

Figure 4. A spatial distribution of vertex degree for the 2D case when $p_{\mathrm{d}} = 0$. Eight walkers started from the origin and 20 000 time steps have passed.

formed on the two-dimensional lattice when $p_{\mathrm{d}} = 0$. Following the next section, changes in the network structure corresponding to the increase of p_{d} are explained.

3. The two-dimensional case without extinction of edges

All vertices in a two-dimensional (2D) squared lattice have degree 4 from the beginning. But vertices that we are interested in are those with degree greater than or equal to 5 made by a random walker's passage. We call vertices that have edges created by random walker's passage, 'vertices with created edges' in the following discussion.

The movement of random walkers in a 2D lattice differs considerably from that in a 1D lattice. Figure 4 shows a spatial distribution of vertex degree observed when $p_{\mathrm{d}} = 0$ and $w = 8$, where w means the number of random walkers. Vertices with large degree are concentrated around the starting point of the random walkers. On the other hand, vertices with small degree are spread on the outside of the degree-rich area. This spatial distribution contradicts that observed for the 1D case, where there is a sharp boundary of the potential well.

Spatial distribution of degree must reflect the increase rate of created edges and vertices with created edges. Figure 5 presents the time dependence of the number of vertices with created edges and of the number of created edges when $p_{\mathrm{d}} = 0$. One can observe in the figure that the number of vertices with created edges increases irregularly but monotonously. This irregular increase of the number corresponds to the accidental escape of random walkers from the potential well, and the slow-down in the increase corresponds to the wandering of random walkers inside the potential well. This behavior can be understood by the smallness of the vertex degrees near the boundary of the potential well, for small vertex degree on the boundary means a few number of edges incident to vertices inside the potential well.

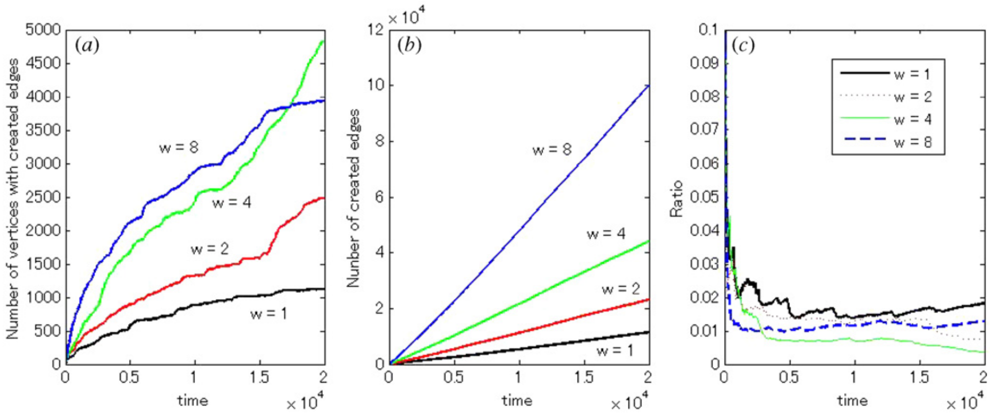

Figure 5. Time evolution of the network in a 2D squared lattice when p_d is 0. (*a*) Changes of the number of vertices with created edges. Each plot corresponds to the number of random walkers, $w = 1$, $w = 2$, $w = 4$ and $w = 8$, respectively. (*b*) Changes of the number of created edges corresponding to (*a*). (*c*) Time dependence of the ratio $2E/(M(M-1))$ where E means the number of created edges and M means the number of vertices with created edges. Note that $M(M-1)/2$ means the maximum number of edges in a graph including M vertices.

On the other hand, the number of created edges indicates a clear linear dependence on time (see figure 5(*b*)). This result also contrasts with the 1D case where the increase rate of new edges becomes small with time. The reason for this slow-down in the increase rate of edges for the 1D case is because new edges can be created only at the boundary of the nearly complete subgraph. As the size of the complete subgraph increases, it becomes difficult for the random walkers to visit the boundary in 1D cases.

In order to consider whether or not the number of edges will be saturated in the potential well in future as seen in the 1D case, the ratio $2E/[M(M-1)]$ is plotted in figure 5, where E means the number of created edges and M means the number of vertices with created edges. As far as the time interval in the figure ($0 < t < 20\,000$) is concerned, the ratio shows up-and-down motion below values of 0.02. This behavior implies that even if several random walkers are added, it is difficult to make the ratio 1, indicating a nearly complete subgraph.

As figure 4 implies extreme inequality of edges for vertices, degree distributions for $p_d = 0$ presented in figure 6 exhibit clear power law $\sim k^{-\gamma}$, where k means degree. As shown in the figure, the exponent γ is set within the range $\gamma \sim 1$–2, which is dependent on the accidental movements of random walkers. The power law is probably supported by the frequent creation of edges around the origin by the return of random walkers and the increase in vertices with small degree caused by the easy escape of random walkers from the potential well. This effect cannot be found in 1D cases.

4. Maintenance of a nearly complete subgraph for a small p_d

In the 1D case, if the extinction probability p_d is sufficiently small, the nearly complete subgraph explained in section 2 is maintained to some extent. In order to know the rate of spread of the nearly complete subgraphs, it is useful to consider the case for $p_d = 0$. Let the number of vertices that constitute the nearly complete subgraph be M, and the number of walkers be w. If a nearly complete subgraph is formed at a given time, the existence probability of a random walker at two end points of the nearly complete subgraph is $2w/M$.

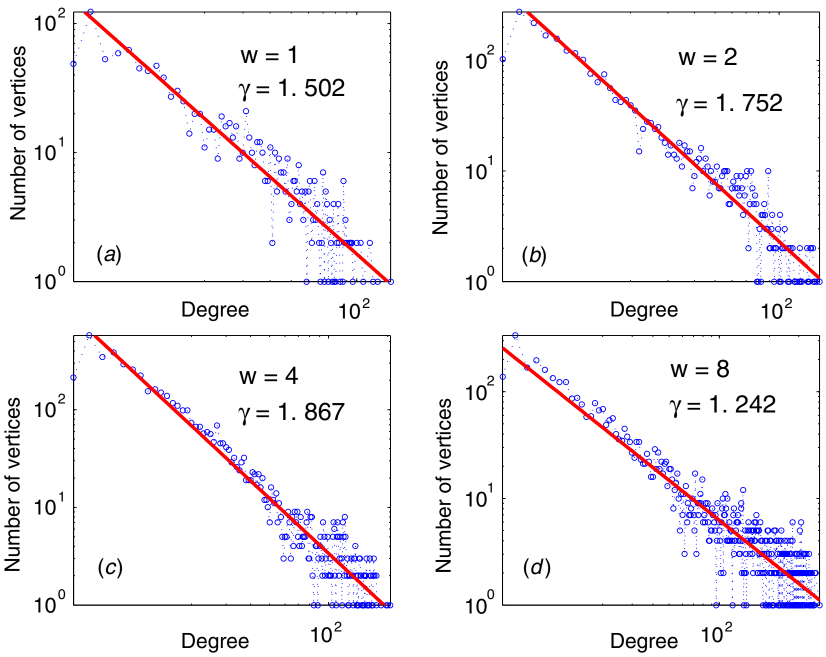

Figure 6. Log–log plots of degree distribution at $t = 20\,000$ time steps for various numbers of walkers when $p_d = 0$. (*a*) $w = 1$, (*b*) $w = 2$, (*c*) $w = 4$, (*d*) $w = 8$. The function $\sim k^{-\gamma}$ where k means degree is fitted to the observed data.

Then the number of new edges created per unit time is roughly estimated as $2w/M$, because the random walkers at the end points may create new edges owing to the incompleteness of edges at the end vertex. On the other hand, the number of new edges created per unit time can be estimated by $M\,\mathrm{d}M/\mathrm{d}t$ because vertices newly added to the nearly complete subgraph at the end points per unit time must join other M vertices to maintain the nearly complete subgraph, and the number of vertices newly added to M per time is $\mathrm{d}M/\mathrm{d}t$. (The increase rate of M is so slow that differentiation of M can be used appropriately.) Therefore we can derive the next formula which explains the time evolution of a nearly complete subgraph,

$$M\frac{\mathrm{d}M}{\mathrm{d}t} = A\frac{w}{M^\alpha}. \tag{1}$$

Here we have described the departure from the rough estimation $2w/M$ by introducing parameters A and α. Note that equation (1) is valid only when the network evolves maintaining nearly complete subgraph.

In figure 7, actual time dependence of the number of vertices that constitute a nearly complete subgraph is compared with the solution of equation (1) $M = [wA(\alpha+2)]^{1/(\alpha+2)}\,t^{1/(\alpha+2)}$. The figure shows that the time evolution is well described by adjusting the parameters and an easy selection of the parameters like $A = 2, \alpha = 1$ yields slightly smaller values than the actual number. But the difference may not be as great, considering that 200 000 time steps have passed. At least the time dependence $M \sim t^{1/3}$ has fairly good agreement with the observation.

When p_d has a positive value, extinction of edges occurs inside the nearly complete subgraph. Let the sum of all strengths on edges in the complete subgraph be 'R'. Since the addition of R per unit time $2w$ is always larger than the new creation of edges in the

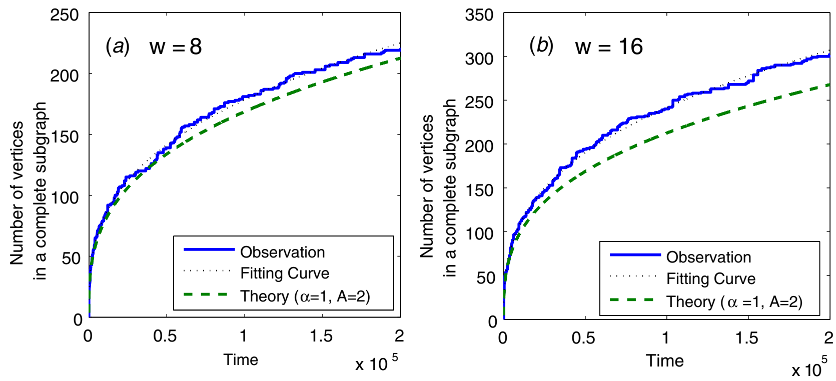

Figure 7. Time dependence of the number of vertices that constitute a nearly complete subgraph for the 1D case when $p_{\rm d} = 0$, (*a*) $w = 8$ and (*b*) $w = 16$. The parameters are calculated as $A = 2.24$, $\alpha = 0.99$ in (*a*) and $A = 1.67$, $\alpha = 0.89$ in (*b*).

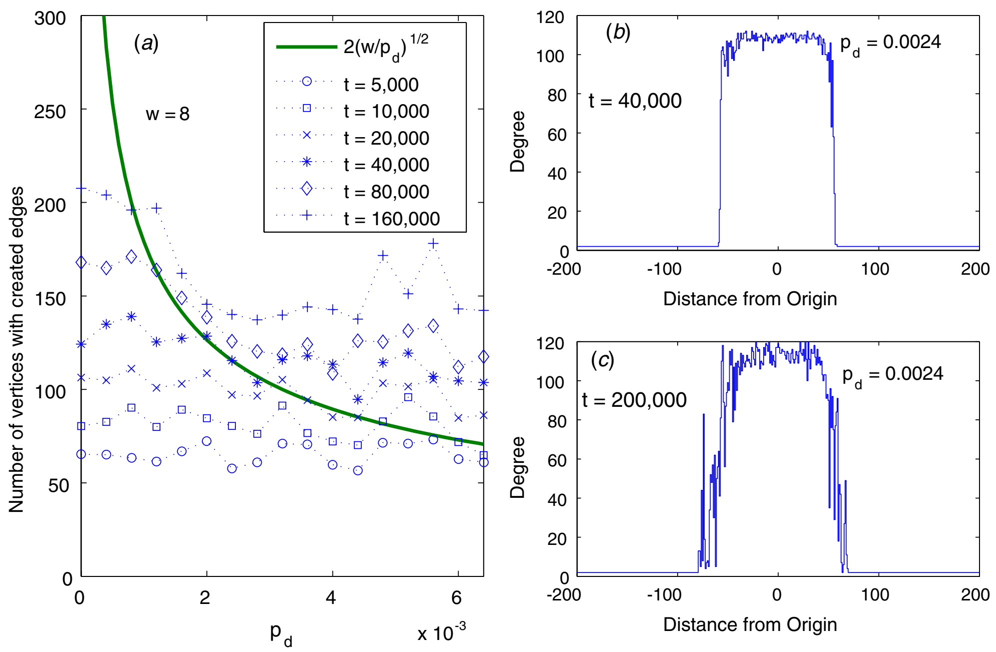

Figure 8. Time evolution of a nearly complete subgraph for small $p_{\rm d}$. (*a*) Number of vertices with created edges M versus extinction probability $p_{\rm d}$ for each time. Spatial degree distribution for $p_{\rm d} = 0.0024$, (*b*) $t = 40\,000$ and (*c*) $t = 20\,0000$.

complete subgraph per unit time, $2w$ must be at least larger than subtraction of R per unit time ${}_MC_2 p_{\rm d} = M(M-1)p_{\rm d}/2 \simeq M^2 p_{\rm d}/2$ to keep the nearly complete subgraph. Therefore the necessary condition for maintaining a nearly complete subgraph is

$$M < 2(w/p_{\rm d})^{1/2}. \tag{2}$$

In figure 8(*a*), plots of M versus $p_{\rm d}$ are presented for each time $t = 5\,000$, $10\,000$, $20\,000$, $40\,000$, $80\,000$ and $160\,000$. This figure shows that, as long as condition (2) is valid, the time evolution of a nearly complete subgraph does not depend on values of p_{d} as much. In other words, time evolution even when p_{d} has a finite value can be described by equation (1). However, figure 8(*a*) shows that the increase rate of M is considerably suppressed after M becomes large enough to break condition (2). One example of such a slow-down is indicated in figures 8(*b*) and (*c*), where the nearly complete subgraph formed at $t = 40\,000$ was hardly damaged after 160 000 time steps.

The slow-down of the increase rate of M can be explained intuitively by considering balances between the creation and extinction of edges at the boundary of nearly complete subgraphs. The mean decrease rate of edges is given by a product of number of edges with strength 1 and p_{d}, so that the mean decrease rate of edges at the two end points of a nearly complete subgraph can be expressed as $2BMp_{\mathrm{d}}$ where B means the ratio of number of edges with strength 1 at the end vertices of the subgraph to M. Since the creation of edges at the end vertices is expressed by the right-hand side of (1), we can derive the condition for the slow-down of the spread of a nearly complete subgraph as the formula $Aw/M^{\alpha} < 2BMp_{\mathrm{d}}$. By substitution of 2 and 1 for A and α,

$$M > (w/Bp_{\mathrm{d}})^{1/2} \tag{3}$$

is obtained as the condition for the slow-down in the spread. Although B is an unknown positive parameter up to 1, it should be noted that there is no difference between notations on the right-hand side of (2) and (3) especially when B is 0.25. Condition (3) should be satisfied, that means B should be over 0.25, when the nearly complete subgraph is stretched as far as possible, because most of strength of edges is expected to decrease to about 1 in such a stretched nearly complete subgraph. However, this discussion cannot be applicable after the nearly complete subgraph is considerably damaged. As p_{d} tends to be large, nearly complete subgraphs are damaged easily after the breaking of condition (2).

5. Maintenance of the power-law distribution for a small p_{d} in the 2D case

A balance of new creation and extinction of edges naturally leads to the stationary size of the formed subgraph. In figures 9(*a*) and (*b*), the number of vertices with created edges M and the number of created edges E on a 2D lattice when $p_{\mathrm{d}} = 0.0004$ are illustrated for various numbers of random walkers w. It is observed that the number of created edges is roughly proportional to w (figure 9(*b*)) regardless of time. But the number of vertices with created edges is not proportional to w especially in the initial time interval (figure 9(*a*)). The departure from the proportionality can be explained intuitively by the cohesion behavior of random walkers at an early stage, which is illustrated in figure 9(*c*). In the figure, six out of the eight walkers were still wandering in an area including the origin, although 40 000 time steps have passed. This cohesion behavior can be understood by the fact that a walker is able to move through edges created by another walker's movement. This multiplier effect is expected to strengthen as w increases. The resulting network is illustrated in figure 9(*d*). It should be noted that created edges can remain only in the area where the walkers stayed recently. In other words, edges cannot survive in the area where the walkers have left. Therefore figures 9(*c*) and (*d*) are similar to each other.

Even when w is large, walkers starting from the same point begin to disperse after a considerable time. So the observation of networks formed by one walker is useful even when discussing network formation by a number of walkers. Figure 10 concerns the time evolution of the network formed by one walker. One characteristic structure is the existence of highly connected vertices joining each other in a certain area like a community. Figure 10(*a*) presents such a cluster constructed by the highest 3% degree in all vertices with created edges. It

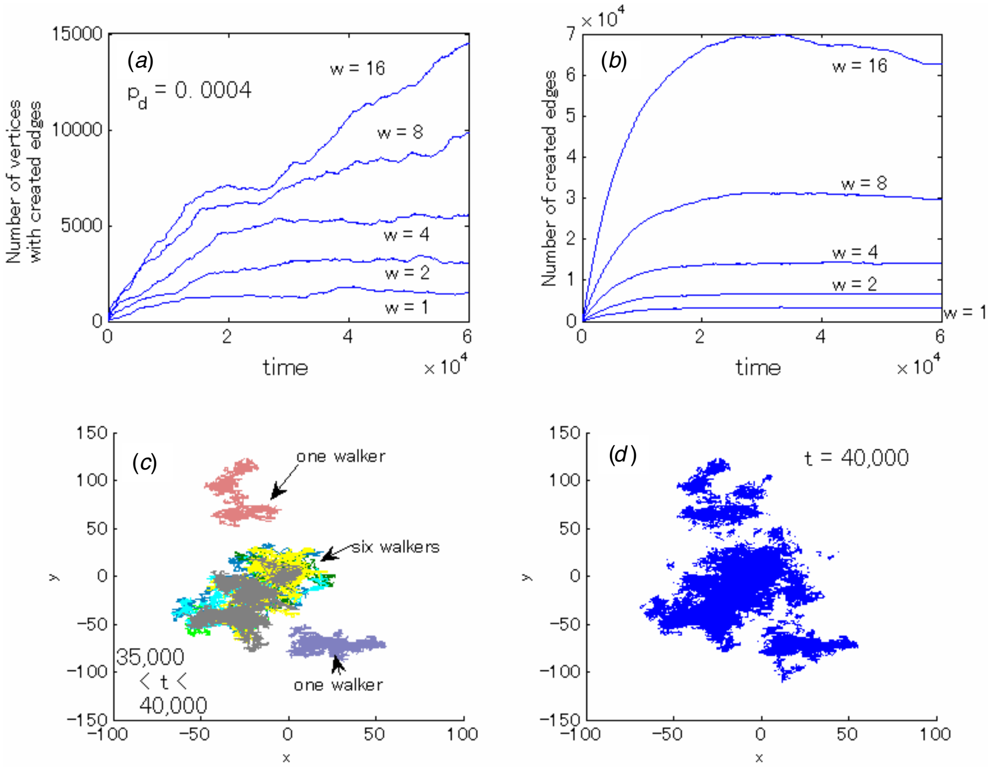

Figure 9. Time evolution of networks on a 2D lattice when $p_d = 0.0004$. (*a*) Changes in the number of vertices with created edges. Each plot corresponds to the number of random walkers, $w = 1$, $w = 2$, $w = 4$, $w = 8$ and $w = 16$, respectively. (*b*) Changes in the number of created edges. (*c*) Traces of eight random walkers observed in a time interval $35\,000 < t < 40\,000$. (*d*) The resulting network formed at $t = 40\,000$.

should also be observed that the cluster has several edges incident to the outside, which bound random walkers to the subgraph. Note that without the frequent arrival of random walkers, the highly connected cluster disappears due to the continuous extinction of edges. In figure 10(*c*) the vertices with the highest 3% degree at $t = 20\,000$ can keep their edges in about one thousand time steps, and sometimes can recover edges in several thousand time steps. But the cluster was found to lose edges without recovering after a certain period passed. Within the period of disappearance of the degree-rich area created at $t = 20\,000$, a new cluster of highly connected vertices came into existence as indicated in figure 10(*b*), although they will disappear like the previous one.

The degree distribution for this case exhibits a power law with an exponent of about 3 (figure 10(*d*)) indicating faster decay than for the case of $p_d = 0$. Large degree in the distribution is supported by the cluster of highly connected vertices. The capacity of the cluster to keep edges and random walkers obviously determines the stability of highly connected vertices. So it is natural that the power-law exponent becomes larger than the case of $p_d = 0$ where degree-rich areas can survive permanently.

For cases with more than one random walker, the network evolution consists of some stages. In the first stage where the number of created edges increases with time (see figure 9(*b*)), the degree distribution exhibits a power law with exponent about 2 (figure 11(*a*))

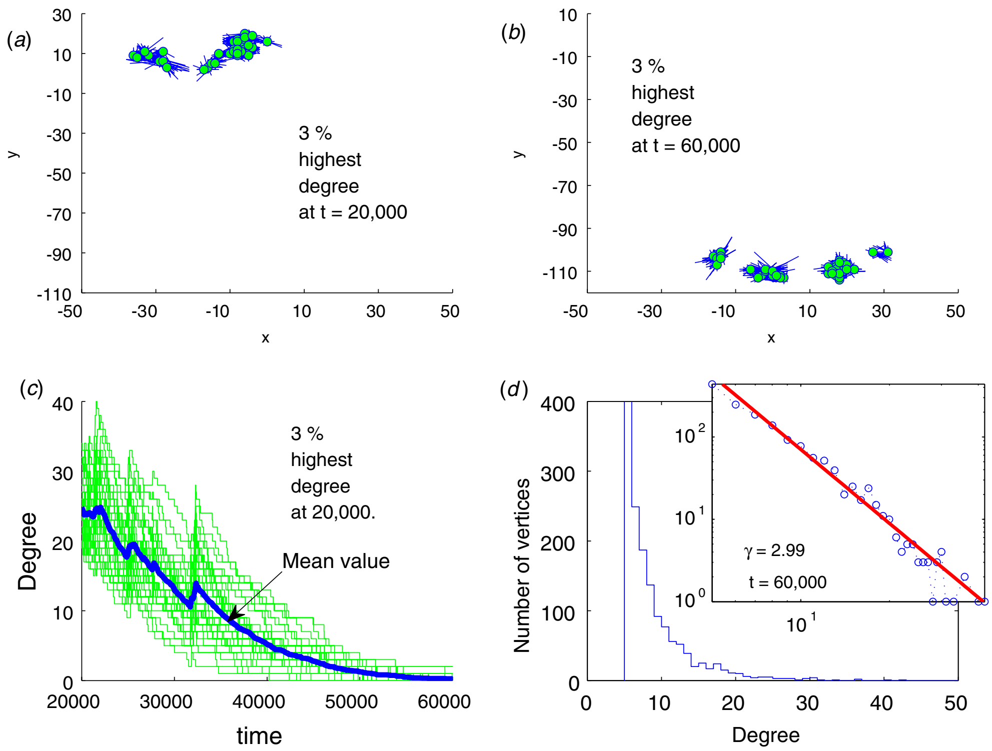

Figure 10. Vertices with the 3% highest degree of the subgraph formed by one walker when $p_d = 0.0004$, (*a*) at $t = 20\,000$ and (*b*) at $t = 60\,000$. (*c*) Time dependence of degrees of these vertices at $t = 20\,000$. (*d*) Degree distribution by one walker at $t = 60\,000$. The observed data are applied to the function $\sim k^{-\gamma}$.

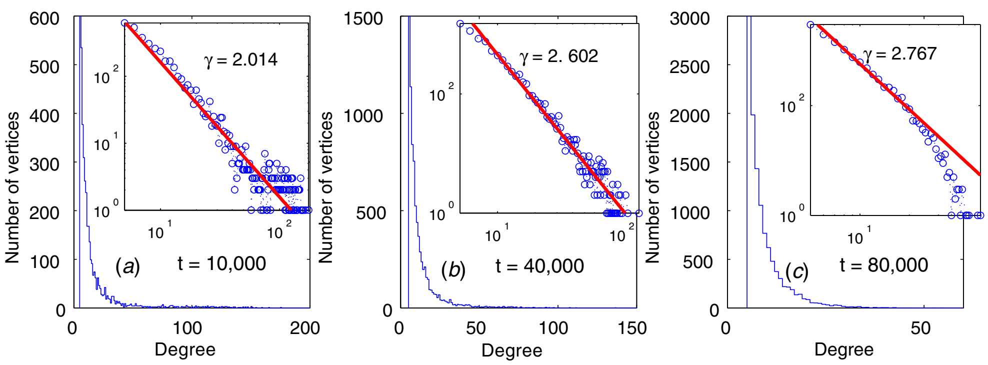

Figure 11. Degree distributions realized by eight random walkers when $p_d = 0.0004$ for each stage. (*a*) $t = 10\,000$ when the number of created edges is still increasing. (*b*) $t = 40\,000$ when some walkers begin to leave an area including the origin. (*c*) $t = 80\,000$ when most walkers have left from an area including the origin.

which has no considerable difference from the cases of $p_d = 0$. This is because all walkers wander around the starting point. Such similarity in initial time intervals can be seen for the 1D case as shown in the previous section.

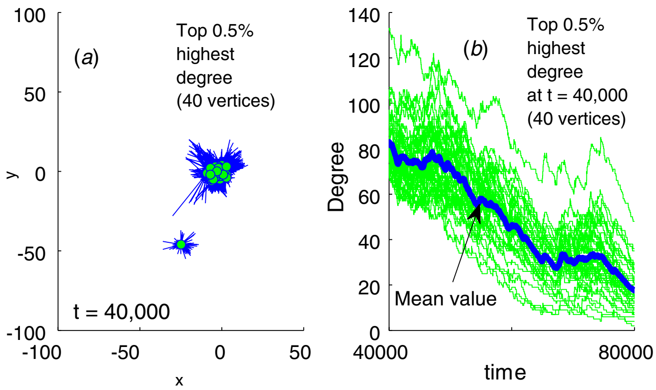

Figure 12. Vertices with the 0.5% highest degree of the subgraph formed by eight walkers when $p_\mathrm{d} = 0.0004$. (*a*) $t = 40\,000$. (*b*) Time dependence of degrees of these vertices. (*c*) Vertices with the 0.5% highest degree at $t = 80\,000$.

In the next stage, the cluster of highly connected vertices around the origin begins to collapse because of the limitation of increase of edges (see figure 9(*b*)) by the extinction of edges and the diffusion of random walkers. The behavior of the collapse is illustrated in figure 12. At $t = 40\,000$, despite two walkers leaving the origin (see figures 9(*c*) and (*d*)), vertices with the highest 0.5% degree still remain around the origin (figure 12(*a*)). The cluster contributes to the existence of degree exceeding one hundred in figure 11(*b*). However, several tens of thousands of time steps later, the cluster loses most of its edges, and vertices with the highest 0.5% degree have distributed discretely corresponding to the near-independent movements of random walkers (figure 12(*c*)). At this stage, vertices with more than one hundred degree disappear owing to the disappearance of the cluster of highly connected vertices made by multiple random walkers. Finally, the maximum vertex degree is reduced until the same value of the maximum degree for one walker's case. The degree distribution will therefore be similar to a one-walker case except there is a cut-off of the power-law regime, although slightly larger vertex degree than the one-walker case may exist because of accidental encounter of random walkers.

6. Collapse of the cohesion

Although it is obvious that sufficiently large p_d leads to separation and emigration of formed subgraphs by the near-independent movements of random walkers, it takes a long time generally to observe such a breakup as seen in figures 2(*e*) and (*g*). In this section, in order to examine the process leading to the breakup of connected subgraph including all random walkers, a life time of highly connected vertices and number of vertices within i steps from the highest connected vertex N_i are calculated, where the life time τ is defined by fitting the function $\exp(-t/\tau)$ to the actual time dependence of the mean vertex degree of vertices with a few % largest degree at a given time.

First let us take the 1D case. In figure 13, N_i/M where $i = 1, 2$ and 3 and τ for networks on 1D lattice are plotted with respect to p_d. Note that formed networks under these conditions have broad degree distribution like in figures 3(*b*) and (*c*). Figure 13(*a*) shows that almost all vertices with created edges are included within three steps from the maximum connected vertex and N_i/M's are nearly constant regardless of values of p_d. This structure has the

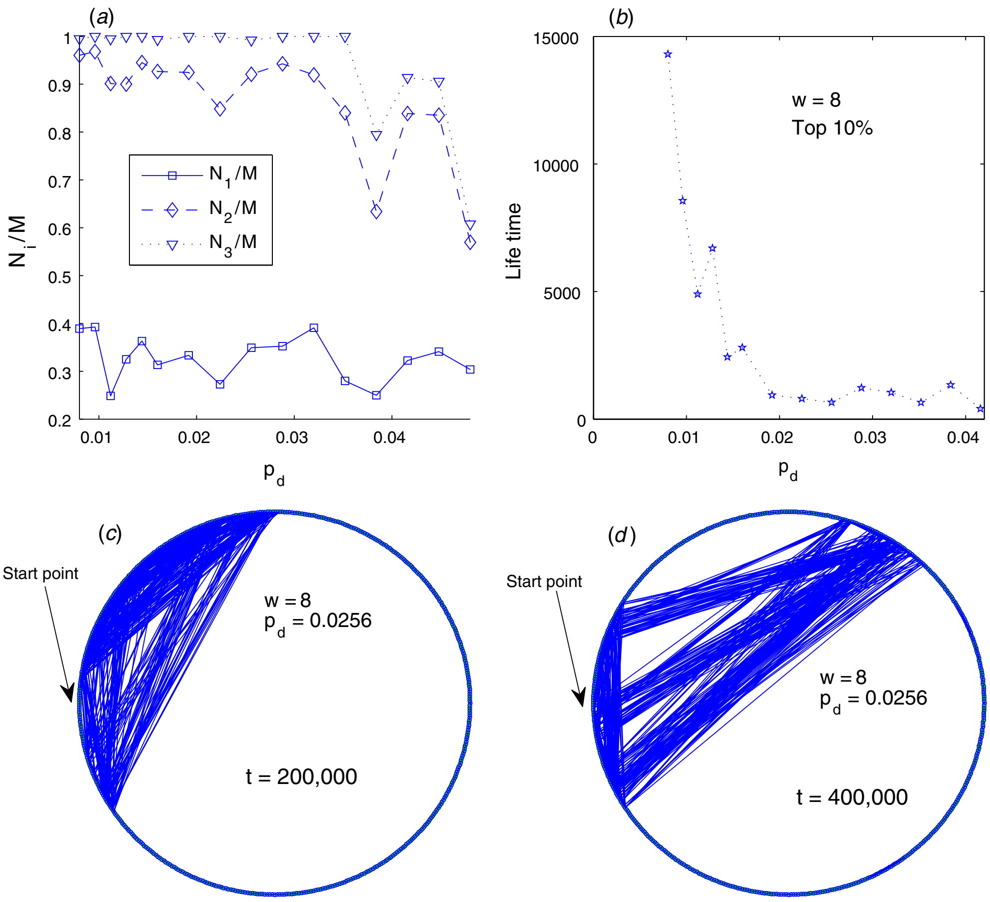

Figure 13. Number of vertices within i steps from a vertex with the largest degree N_i and life time of vertices with large degree τ for a case of $w = 8$ on a 1D lattice. (*a*) Ratio of N_i to the number of vertices with created edges M is plotted for various $p_{\rm d}$. (*b*) The life time τ defined for $180\,000 < t < 20\,000$ versus $p_{\rm d}$. (*c*) A network actually formed for $p_{\rm d} = 0.0256$ at $t = 200\,000$ and (*d*) at 400 000. There are 500 vertices in the ring. (Random walkers have not made around the ring.)

advantage of maintaining all edges in one connected subgraph, because random walkers can easily reach all vertices with created edges and strengthen their edges. On the other hand, figure 13(*b*) shows that τ is easily reduced by an increase of $p_{\rm d}$. These features indicate that there is the rise and fall of edges in a connected subgraph in which all pairs of vertices are connected by a small distance. Figures 13(*c*) and (*d*) show an example of such a connected subgraph at $t = 200\,000$ and at $t = 400\,000$ where the degree-rich area has moved in the time interval. The sudden decrease in N_i/M with respect to $p_{\rm d}$ in figure 13(*a*) corresponds to escapes of random walkers from the potential well (see figure 2(*e*)), while decrease of τ in figure 13(*b*) is considered a premonitory sign of such a breakup of the connected subgraph.

Our calculation also provides the next empirical rule for the number of created edges E after sufficient time has passed,

$$E \simeq (2\sqrt{w/p_{\rm d}})^2/2 = w(2\sqrt{1/p_{\rm d}})^2/2 = 2w/p_{\rm d}. \tag{4}$$

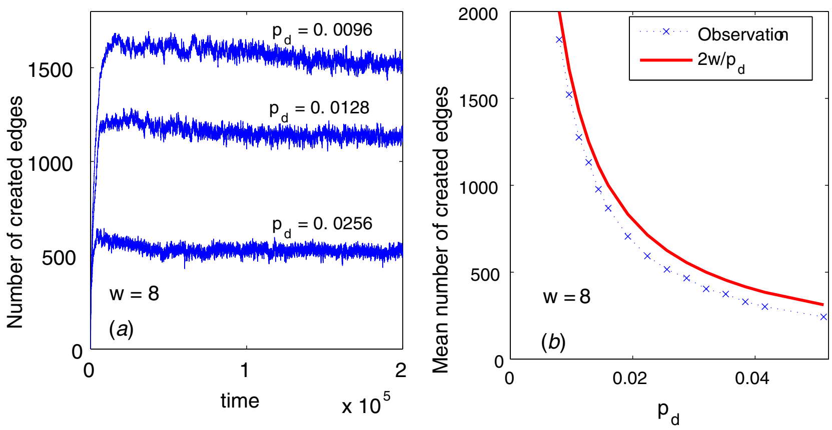

Figure 14. The number of created edges for the 1D case when $w = 8$. (*a*) Typical examples of time dependence of created edges. The number approaches a stationary behavior regardless of p_{d}. (*b*) Mean values of E in $180\,000 < t < 200\,000$ are compared with the empirical rule (4) for various p_{d}.

Time dependences of E are plotted in figure 14(*a*) and the mean value of E in $180\,000 < t < 200\,000$ is compared with the empirical rule (4) for various p_{d} in figure 14(*b*). Figure 14(*b*) shows that the rule (4) is valid especially when p_{d} is small. It should be noted that the value in (4) is equal to the number of edges that constitute the maximum nearly complete subgraph given by equation (2). Therefore figure 14(*a*) indicates a stagnation in the increase of edges after the collapse of nearly complete subgraphs. Furthermore, $2(1/p_{\mathrm{d}})^{1/2}$ in (4) means the maximum number of vertices for maintaining a nearly complete subgraph only by one walker's movement. Therefore the empirical rule (4) also means that the capacity to keep edges by w walkers is given by the sum of the number of edges of a nearly complete subgraph by one walker's movement $2/p_{\mathrm{d}}$. Although a complete subgraph is not actually formed here, walkers can keep a similar capacity to create edges by the cooperation of many walkers in a compact subgraph discussed above. On the other hand, our observation shows that the number of vertices that constitute the connected subgraph can continue increasing after the collapse of nearly complete subgraphs until over $2w(1/p_{\mathrm{d}})^{1/2}$. Increase of p_{d} which decreases $2(1/p_{\mathrm{d}})^{1/2}$ to a small number can be regarded as an approximate criterion for the collapse of a connected subgraph, because small $2(1/p_{\mathrm{d}})^{1/2}$ means reduction of the number of vertices that one walker can influence.

The coefficient of w/p_{d} in equation (4) can be interpreted by the following discussion. The time evolution of E is considered to follow the next equation,

$$\mathrm{d}E/\mathrm{d}t = c_1 w - c_2 p_{\mathrm{d}} E, \tag{5}$$

where c_1 is the probability of the creation of edges by one walker's movement per one time step and c_2 is the ratio of the number of edges of strength 1 to E. Both values fluctuate with time irregularly. But after considerable time has passed, $\mathrm{d}E/\mathrm{d}t$ can be regarded as 0, and c_1 and c_2 as constants by ignoring small fluctuations with time. Therefore E can be roughly estimated as

$$E = c_1 w / c_2 p_{\mathrm{d}}. \tag{6}$$

Although c_1 is 0 in a complete subgraph, c_1 increases with time up to 1 as the mean vertex degree decreases with time. The decrease in the coefficient of w/p_{d} from 2 by the increase in

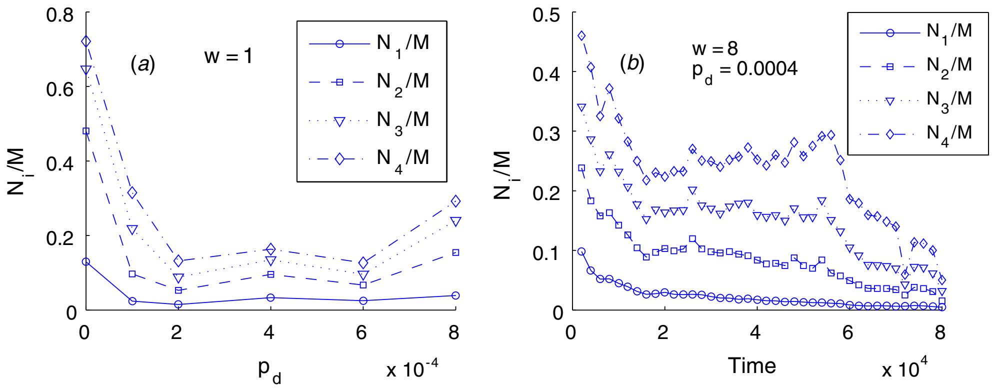

Figure 15. For a case 2D lattice, number of vertices within i steps from a vertex with the largest degree N_i. (a) Ratio of N_i to all the number of vertices with created edges M is plotted for various p_{d} when $w = 1$ and $t = 40\,000$. (b) Time dependence of N_i/M for a network formed by eight walkers in a 2D lattice when $p_{\mathrm{d}} = 0.0004$.

p_{d} shown in figure 14(b) therefore directly means an increase in c_2. Increase of c_2 can also be considered a premonitory sign of the breakup of one connected subgraph, because the increase in c_2 should naturally weaken tolerance to the extinction of edges.

For the 2D case, as mentioned in the previous section, once an edge-rich area has been lost, random walkers cannot come back there easily and the formed subgraph begins to emigrate. The results of the calculation of N_i/M's for the 2D case are consistent with this observation. Figure 15(a) presents changes of N_i/M's at $t = 40\,000$ with respect to various p_{d} for networks formed by one walker in a 2D lattice. This calculation shows that N_4/M is only a 70% even when $p_{\mathrm{d}} = 0$, which means that there is large fraction of vertices far from the vertex with the highest degree. The existence of vertices far from the vertex with the highest degree frees random walkers from highly connected vertices. This result is different from that for 1D cases where all vertices in the subgraph rather than only some of the vertices are sharing all walkers.

Figure 15(b) presents changes of N_i/M's with time for a network formed by eight walkers in a 2D lattice when $p_{\mathrm{d}} = 0.0004$. The figure clearly indicates the process of the network evolution discussed in the preceding section. In the first stage in which all walkers are bound to the origin, N_i/M's are large, but decreases with time ($t < 20\,000$). In the second stage in which the cluster of highly connected vertices around the origin remains, but begins to collapse slowly, N_i/M's keep larger values ($20\,000 < t < 60\,000$) than the values of N_i/M's of one walker. Finally, most walkers leave the origin, and N_i/M's decrease to as small as that of one walker ($60\,000 < t$).

7. Summary and discussion

We have investigated networks formed by a diffusion process of random walkers leaving behind edges according to the rules explained in section 1. Calculations are carried out under the condition that all walkers spread from the same vertex in a 1D or a 2D squared lattice. These initial lattices where the walkers can wander provide a geographical consideration to the network, that is, edges joining nearest-neighbor vertices in the initial lattice will never be extinct and vertices far from each other cannot be joined without transports of random walkers between them. The creation and extinction rule of edges expresses a situation where connections between elements of a system grow or degenerate according to the degree of prosperity of random transports which depart from one point. However, the random transports described by random walkers' movements are extreme in that the transports do not exhibit birth and death, and movements of random walkers are always restricted by the last movement and the local structure around the walker. Prominent suppression of the spread of random walkers at the boundary of a nearly complete subgraph in a 1D lattice is a typical example of the restrictions on movements of random walkers by local structure of networks. As a result, the diffusion process of random walkers and resulting networks greatly depend on the dimension of initial lattice.

For 1D cases, we developed some formulae for describing the time evolution of formed subgraphs. As long as p_{d} is sufficiently small, a nearly complete subgraph is formed by the random walkers starting from the same vertex. The increase of the number of vertices of nearly complete subgraph M can be described by the formula $M = (6wt)^{1/3}$ until M exceeds the value $2(w/p_{\mathrm{d}})^{1/2}$ that means the condition for collapse of nearly complete subgraphs. It takes a very long time to transit from a complete subgraph to a subgraph with broad degree distribution especially when p_{d} is small. Connected subgraphs with broad degree distribution formed after collapse of the complete subgraph keep about $2w/p_{\mathrm{d}}$ edges. The expression $E \sim w/p_{\mathrm{d}}$ can be derived by a simple consideration for the time evolution of a number of created edges, where the coefficient of w/p_{d} can be interpreted as a parameter that shows a premonitory sign of breakup of one connected subgraph. The connected subgraphs can be stabilized by small vertex–vertex distances between all pairs of vertices, because such a structure enables all vertices to keep random walkers without distinction. As p_{d} tends to large, the complete subgraph easily transits to a connected subgraph with broad degree distribution, and shortened life time of highly connected vertices is observed. In this situation, there is a dynamical local rise and fall of edges in the connected subgraphs. Finally, when the value $2(1/p_{\mathrm{d}})^{1/2}$ is a small number, the connected subgraph becomes easy to break and subgraphs begin to emigrate corresponding to near-independent movements of walkers.

For 2D cases, the power-law distribution of vertex degree is observed especially when p_{d} is 0. The power-law exponent γ can take values in the range $1 < \gamma < 2$ corresponding to the unpredictable movements of random walkers. The power law can also be observed for a positive p_{d}. Especially in an early stage of network evolution, there is only a slight difference from the case of $p_{\mathrm{d}} = 0$. However, the exponent increases with time until about 3 even when p_{d} is very small. The formed network structure is characterized by clusters of highly connected vertices which contribute to a large degree region in the degree distribution. When p_{d} is 0, the cluster of highly connected vertices can survive permanently. But when p_{d} is not 0, the cluster of highly connected vertices collapses over a long time. Collapse of cluster of highly connected vertices must be related to the fact that the number of vertices that can be reached from the highest connected vertex in a few steps is only limited in several tens of per cent of all vertices with created edges. As a result, some vertices with large degree allow random walkers to escape to vertices with small degree, while in the 1D case, walkers-sharing in all vertices with created edges controls the dispersion of walkers. Although the addition of random walkers can somewhat stabilize the cluster of highly connected vertices by increasing of N_i/M, the breaking out of the cluster is beyond repair.

It should be reaffirmed that our calculation was carried out for cases where all random walkers start from the same vertex. It is unlikely that random walkers far from each other spontaneously gather and form subgraphs reported in this paper. In this sense, some networks observed in this paper can be considered to be in a kind of metastable state. A case examined in our previous paper [21] where 100 walkers were confined in 500 aliened vertices is considered as a study for dense random walkers, since 100 walkers can spread in a connected subgraph wider than 500 vertices.

A common interpretation between the 1D and 2D cases is that the formed network is a result of the statistics of the past movements of random walkers although only a few number of walkers can be observed at one time. As a result, an edge-rich area where the random walkers wander to can be easily created in both dimensions. However, only the random removal of edges from the edge-rich area will never lead to the results obtained in this paper, for random removal of edges from a nearly complete graph is equivalent to the random linking of vertices without edges. Random transportation which accompanies the new creation of edges is indispensable for acquiring a broad degree distribution, since random walkers tend to provide new edges to edge-rich areas more than to edge-poor areas. Random walkers in 2D cases have another feature where they can sometimes leave edge-rich areas even when $p_{\mathrm{d}} = 0$.

More realistic description of transports in networks may be a theme to be studied in the future, for the transports described by random walkers are extreme concepts as previously mentioned. The proposal of other transportation models may uncover other features of network evolution.

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